CALCULATION OF DEFLECTION OF THE VERTICAL COMPONENTS: ANALYZING THE GPS, LEVELLING MEASUREMENTS AND THEIR DISTRIBUTION GEOMETRY

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ABSTRACT: Precise calculation of survey networks, such as reduction of directions, azimuths, zenith angles, slope distances onto the ellipsoid and geoid modeling processes are topically demanded applications of deflection of vertical. Astro-geodetic and gravimetric techniques, incorporated with GPS data and terrestrial data, are complex and time consuming methods of determining this value. This study resolved the deflection of the vertical components in north-south (ξ) and east-west (η) of a fixed point by using GPS and levelling measurements, and investigated the effect of distribution geometry of the ancillary stations around the selected station for the calculated results. This process was enhanced by considering both spreaded distribution geometry of the stations and skewed distribution geometry of the stations.

The deflection of vertical components were calculated for a fixed point in the area of Sabaragamuwa University of Sri Lanka, using two test networks for both spreaded and skewed distribution geometries with 6 ancillary stations and a test station ranging from 300 to 1200 meters. Geoidal heights and ellipsoidal heights were obtained by using levelling and GPS measurements. General least-squares solution applied to mathematical model that relates the unknown parameters (ξ, η) with GPS and levelling observables. Deflection of vertical components η and ξ of the test station in the spreaded distribution network were η = -2.49086695" ± 4.29708426" and ξ = -25.08863327"± 3.74482408". While the components of the same station in the skewed distribution network were η = -10.57460420" ± 6.16231994"and ξ = -23.58654648" ± 1.28286691".

Even though the north-south component was found to be approximately consistent in the two networks, the east-west component deviated up to some extent. Nevertheless, magnitude of deflection of the vertical (ε) obtained from spreaded distribution network (ε = 25.21198004" ± 3.90504970") was not considerably deviated from the value obtained from skewed distribution network (ε = 25.84854791" ± 3.00577546"). Finally, the results were compared with EGM2008 derived deflection of the vertical components.

1. INTRODUCTION

Due to the complexity of the physical earth surface, it is impossible to approximate the shape of the Earth with any reasonably simple mathematical model. Hence the measurements have typically substituted into simpler surfaces for easy measurements and calculations. The Geoid is a model of the Earth's surface that represents the mean global sea level. An ellipsoid is the mathematical reference of the Earth on which the geoid is represented. The difference between these two surfaces is called geoid undulation (N). The angular difference between the direction of the gravity vector at a point on the geoid and the corresponding ellipsoidal normal through the same point for a particular ellipsoid is called deflection of the vertical (θ) (Featherstone, 1999).

Since the plumb lines are orthogonal to the level surfaces by definition, the deflection of the vertical also gives a measure of the gradient of the level surfaces with respect to a particular ellipsoid. A north-south or meridional component (ξ) and an east-west or prime vertical component (η) are the usually decomposed two mutually perpendicular components of the deflection of the vertical. Deflection components are positive if the direction of the gravity vector points further south and further west than the corresponding ellipsoidal normal (Vanicek and Krakiwsky, 1986).

Historically, deflection of the vertical is obtained by Astro-Geodetic and gravimetric techniques. These techniques are very complex and time-consuming. Also, it is not a typical land surveyor problem until the recent introduction of Satellite Positioning Techniques (GNSS) such as GPS in geodesy. Because of global positioning system (GPS) and leveling measurements contain information about the ellipsoidal height and orthometric height, respectively, they can be used to determine deflection of the vertical components. In the past, several investigations by Soler et al. (1989), Vandenberg (1999), Fujii (1990), and Evans et al. (1989) show that the calculation of deflection of verticals with the amalgamation of these quantities is possible. For the first time, Soler et al. (1989) Compared vertical deflections determined by GPS and spirit leveling with classical Astro geodetic deflections.

Deflection of the vertical components of a station located in Hong Kong were estimated from the recent studies by Tse and Baki Iz (2006). The agreement of the deflection of the vertical components ξ, and η obtained from the experiment were -7".3 ± 1."6 and 5".3 ± 4".3 respectively. It has been noticed that the north-south component approximately consistent, while east-west component differs from some extent. According to the study, the reason for that is due to the distribution geometry of the ancillary stations around the selected station. Therefore, aoording to the suggestions given by the previous method (Tse and Bâki, 2006), it is worth to calculate the deflection of vertical components at the fixed point by considering both spreaded and skewed distribution geometry of the stations. This present study tests the method of determining the deflection of the vertical components from GPS and leveling measurements considering the distribution geometry. In this method, geoid heights and ellipsoidal heights are found by using leveling and GPS measurements respectively. The deflection of vertical components are calculated for a fixed point in the area of Sabaragamuwa University of Sri Lanka.

2. DEVELOPMENT OF THE MODEL

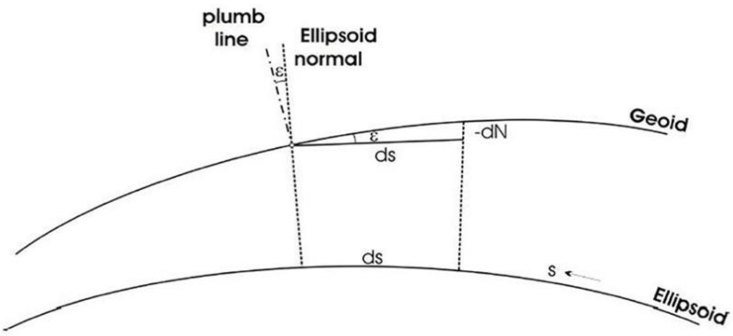

Figure 1: Relationship between geoid height and deflection of the vertical (Heiskanen and Moritz, 1967)

The fundamental aim of developing a mathematical model is to establish a relationship between the unknown parameters (η, ξ) and the GPS and leveling observables. Vertical deflection is found by differencing the geoidal and ellipsoidal heights. Knowing the height of the ellipsoid and the height of the geoid at the solution station, the normals to both surfaces can be found. Their difference is the deflection of the vertical (ε). The relationship between geoid height and deflection of the vertical is defined through the following formulae (Heiskanen and Moritz, 1967).

$$dN = -\,\varepsilon\,.\,ds \tag{1}$$

$$\varepsilon = -\frac{dN}{ds} \tag{2}$$

Deflection of the vertical in any geodetic azimuth (α) direction can be calculated as follows, by using north-south and east-west components (η, ξ).

$$\varepsilon = \xi\, cos\alpha + \eta\, sin\alpha \tag{3}$$

Combining (2) and (3) equations,

$$-\frac{dN}{ds} = \xi\, cos\alpha + \eta\, sin\alpha \tag{4}$$

If replacing the differential elements that appears in the above relationships with their discrete counterparts,

$$-\frac{\Delta N}{\Delta s} \approx \xi\, cos\alpha + \eta\, sin\alpha \tag{5}$$

When considering the geoid-ellipsoid separations (geoid undulations) at two closely spaced locations A and B on the surface of the earth, geoid heights can be defined in terms of ellipsoidal (h) and orthometric heights (H) by using the following equations:

$$N_A = h_A - H_A \tag{6}$$

$$N_B = h_B - H_B \tag{7}$$

By subtracting equation (7) from equation (6) gives the geoid height difference (ΔN_{AB}) between point A and point B as follows:

$$\Delta N_{AB} = N_A - N_B = (h_A - h_B) - (H_A - H_B) = \Delta h_{AB} - \Delta H_{AB} \tag{8}$$

Substituting the above expression in equation (5),

$$-\frac{\Delta h_{AB} - \Delta H_{AB}}{\Delta S_{AB}} \approx \xi \cos \alpha_{AB} + \eta \sin \alpha_{AB} \tag{9}$$

Here, ΔH refers to values from geometric leveling and Δh refers to the values obtained from GPS measurements. In this case, equation (9) is a two-variable equation ($\xi\ and\ \eta$).

In this equation, azimuth (α) can be calculated through geodetic coordinates measured in points A and B. In order to calculate the deflection of the vertical components for any point A, one requires secondary points, such as B and C. Deflection of the vertical components of point A can be calculated using the ellipsoidal and orthometric heights of point pairs of (A,B) and (A,C). In addition, it is also possible to calculate the deflection of the vertical components by solving the deflection of the vertical components of any point with the help of the values pertaining to three or more points distributed around the selected point, and by using a general least-squares solution (Mikhail 1976).

It is important to model the errors to decide whether the model is feasible. Some of the levelling errors are systematic and can be eliminated; other errors maybe estimated form the least-squares error propagation model.

It is important, that the above relationships hold if the separations between the master and the ancillary stations are very small. Nevertheless, the larger the separation between two stations, the smaller the error of the deflection of the vertical computed from the height differences as shown below.

Considering the left hand side of (9),

$$\varepsilon \approx -\frac{\Delta h - \Delta H}{\Delta s} \tag{10}$$

Using the variance propagation rule and assuming that orthometric height differences and ellipsoidal height differences are not correlated,

$$\sigma_\varepsilon{}^2 = \frac{1}{\Delta s^2}(\sigma_{\Delta H}^2 + \sigma_{\Delta h}^2) + \left(\frac{\Delta h - \Delta H}{\Delta s^2}\right)^2 \sigma_{\Delta s}^2 \tag{11}$$

The second term within parentheses in the above expression is a fourth order term and can be safely omitted. Hence,

$$\sigma_\varepsilon{}^2 = \frac{1}{\Delta s^2}(\sigma_{\Delta H}^2 + \sigma_{\Delta h}^2) \tag{12}$$

Following table include the theoretical standard deviation values.

Aoording to Ceylan (2009), the error in the deflection of the vertical is linearly proportional to the errors of GPS and leveling measurements. For closely spaced stations, both measurement techniques can be very accurate because the systematic errors cancel out for GPS height differences and do not accumulate for the leveling observations.

3. NETWORK GEOMETRY

Two test networks were developed suitable for both leveling and GPS observations around the Sabaragamuwa University, from latitude $6^0\ 42^{'}\ 7.80^{'}$ to $6^0\ 43^{'}\ 20^{'}$ and from longitude $80^0\ 46^{'}40^{'}$ to $80^0\ 47^{'}\ 38^{'}$. Each network consists of six ancillary stations and the test station. The positions of the ancillary stations in the spreaded distributed network were established as the ancillary stations spread around the test station, while most of the ancillary stations in the skewed distributed network were established in the north-south direction (Figure 2).

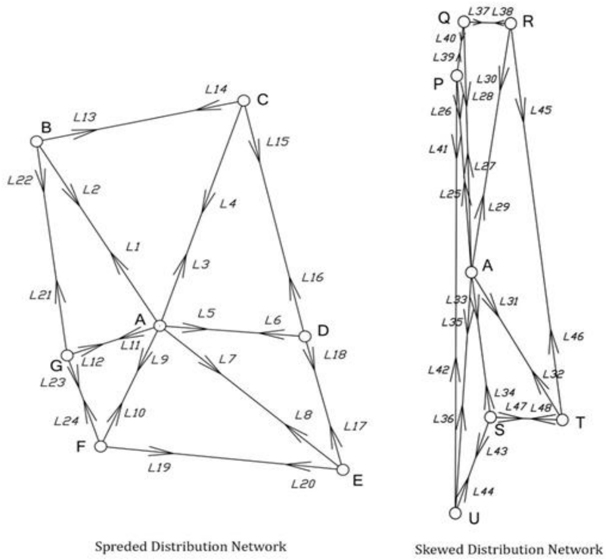

Figure 2: Leveling routes between established stations

Field works were carried out to determined orthometric heights of all ancillary stations in the designed level networks. Double run second order differential leveling was run for twenty-four leveling routes between established stations in both networks and 48 height difference measurements were obtained. Scheduled leveling routes between established stations shown in the Figure 2. The leveling misclosures between double-run level lines were within the tolerance level for tertiary leveling ($8\sqrt{k}$ mm where the k is the distance between terminals in kilometers). Observations were adjusted using the least square adjustment method. Two Leica and Trimble GPS systems were used for the GPS observations in static mode. GPS observations were post processed and adjusted by baseline processing using two reference stations (NSG 01 and NSG 03).

4. COMPUTATIONS

Equation 9 includes two components of the deflection of vertical as unknown parameters which can be estimated from orthometric and ellipsoidal height differences and adjusted using the general least squares method as below (Mikhail, 1976).

$$l = f(x) \tag{13}$$

And nonlinear conditions in the model can be expressed functionally by,

$$f(l) = 0 \tag{14}$$

This can have a more general case in which several observed quantities and unknown parameters are in the same equation. In that case, a mathematical model has the following form since the function is made of c equations, *x* of *u* unknowns and *l* of *n* observations.

$$f(l, x) = 0 \tag{15}$$

In general, the conditional as well as the constraint equations involved in an adjustment problem can be nonlinear. However, least squares treatment is generally performed with linear functions, since it is rather difficult and often impractical to seek a least squares solution of nonlinear equations. Consequently, whenever the equations in the model are originally nonlinear, Series expansions and Taylor's series in particular are often used to get linear equations.

$$A\, v + B\ \Delta = f \tag{16}$$

It is assumed that the orthometric and ellipsoidal height observations are uncorrelated. Standard deviations of orthometric heights and ellipsoidal heights are used for the derivation of variance-covariance matrix. Then full variance-covariance matrix used for the derivation of weight matrix.

Equation (16) is the fundamental form of condition equations for the adjustment of observations and independent parameters combined. It represents c linear equations in $(n + u)$ unknowns, which are the elements of the two vectors $\boldsymbol{v}$ **(**The residual of observations**)** and Δ **(**Respective corrections for unknown parameters denoted by *x***).** A unique least squares solution is obtained by adding the basic criterion of the following equation.

$$\phi = v^t W v \qquad \text{Should be a minimum} \tag{17}$$

To enforce this criterion and at the same time have a solution to the equation (16), the method of constrained minima of Lagrange multipliers is used. Thus, if $k_{(c,1)}$ represents the yet unknown Lagrange multipliers, then should seek the minimum of the following function, noting that the quantity between parenthesis is zero when equation (16) is satisfied.

$$\phi' = v^t W v - 2k^t(A v + B \ \Delta - f) \tag{18}$$

To minimize ϕ', its partial derivatives with respect to $\boldsymbol{v}$ and to Δ are equated to zero. Realizing of course that W is a symmetric matrix.

$$-W_{(n,n)} v_{(n,1)} + A^t{}_{(n,c)} k_{(c,1)} = 0 \tag{19}$$

$$B^t{}_{(u,c)} \ k_{(c,1)} = 0 \tag{20}$$

In matrix form the total system is,

$$\begin{bmatrix} -W & A^t & 0 \\ A & 0 & B \\ 0 & B^t & 0 \end{bmatrix} \begin{bmatrix} v \\ k \\ \Delta \end{bmatrix} = \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix} \tag{21}$$

This system of equations has usually been referred to as the total system of normal equations. The matrix of coefficients is a square symmetric matrix of order $(n + u + c)$, which is always nonsingular (that is, its rank is equal to its order), unless the model is improperly constructed. In view of this fact the least squares problem can be solved by inverting the system of equation (21). Furthermore, both $\boldsymbol{v}$ and Δ may not be interested in, together but only in one of them, and rarely need k for its own sake. Henceforth, an alternative scheme may be desirable. Fortunately, the system of equation (18) contains many zero sub matrices and a solution by partitioning is relatively simple. From equation (19),

$$v = W^{-1} A^t k = Q A^t k \tag{22}$$

And substituting in equation (16) gives

$$A\, Q A^t k + B \ \Delta = f \tag{23}$$

By applying the propagation rule,

$$[B^t \,(A Q A^t)^{-1} B \,] \,\Delta = [B^t \,(A Q A^t)^{-1} f] \tag{24}$$

Equation (24) represent a set of *u* equations in *u* unknown parameters (the elements of Δ**)** which are termed partially reduced normal equations.

$$N = B^t W_e B = B^t \,(A Q A^t)^{-1} B \tag{25}$$

$$t = B^t W_e f = B^t \,(A Q A^t)^{-1} f \tag{26}$$

With the above auxiliaries, a more compact form of equation (24) is

$$N \,\Delta = t \tag{27}$$

The vector Δ can be obtained from equation (27) by direct inversion such that

$$\Delta = N^{-1} t \tag{28}$$

In the derivation above, several inverses were taken. First, W^{-1} in equation (22) is all right since W is nonsingular because the observations are functionally independent. Finally, N^{-1} in equation (28) is also possible because N has a rank and order that are equal ($= u$). Having the value of the parameters Δ, the vector k can be computed and substituted into equation (22) to evaluate the vector of residuals v.

Table 1: Calculated values for spreaded network

From	To	Geodetic		Orthometric height differences (m)	Ellipsoidal height differences (m)
		Azimuth (degree)	Distaance (m)		
A	B	327.83358	1031.293	28.960	29.060
	C	18.90971	1121.428	21.070	21.239
	D	93.12071	641.840	-10.330	-10.333
	E	129.06302	1069.217	-50.285	-50.344
	F	205.62790	579.876	-42.942	-42.986
	G	270.91435	385.593	-41.539	-41.486

Table 2: Calculated values for skewed network

From	To	Geodetic		Orthometric height differences (m)	Ellipsoidal height differences (m)
		Azimuth (degree)	Distaance (m)		
A	P	355.98314	874.889	40.342	40.428
	Q	358.41817	1121.421	38.519	38.656
	R	5.46089	1140.798	33.501	33.650
	S	171.11164	613.165	-15.261	-15.328
	T	147.70497	752.247	-36.185	-36.238
	U	185.74166	1049.018	-24.030	-24.136

Table 1 and 2 show the calculated geodetic azimuth and distances, orthometric height differences and ellipsoidal height differences from the solution station to other stations.

5. ESTIMATION OF THE DEFLECTION OF THE VERTICAL COMPONENTS

5.1. SPREADED DISTRIBUTION NETWORK

According to the spreaded distribution network, η and ξ components of the deflection of vertical at the solution station A was found to be - 0° 0' 2.49086695" ± 0° 0' 4.29708426" and - 0° 0' 25.08863327 " ± 0° 0' 3.74482408 " respectively. And magnitude of the deflection of vertical was found to be 0° 0' 25.21198004" ± 0° 0' 3.90504970".

The solution has four degree of freedom due to the six conditioned equations and two unknown parameters. The a priori variance is 1. The a posteriori variance is 0.47812142 and that indicates priori standard deviations are too small and observations have been adjusted more than predicted. Solution passed the two-tailed Chi-Squared Test at the 95% confidence level. Therefore the a priori and a posteriori stochastic models can be considered as valid.

Residual and standard deviation for the orthometric height difference and ellipsoidal height difference were reached to mm level (Figure 3). The following histogram of residuals (Figure 4) suggests that the residuals (and hence the error terms) are normally distributed. All residuals have clustered about the midpoint and fit into the bell shaped curve. And the normal probability plot of the residuals is approximately linear supporting the condition that the error terms are normally distributed.

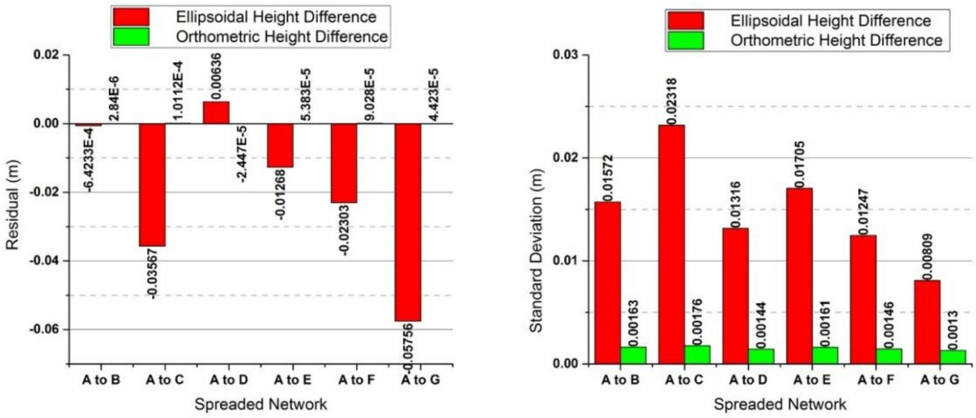

Figure 3: Residual and standard deviation for the Ellipsoidal height difference and Orthometric height difference in Spreaded network

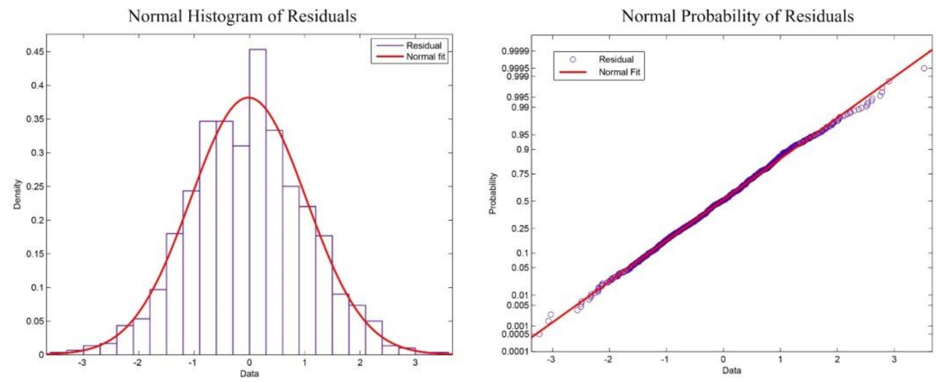

Figure 4: Normal histogram and normal probability of residuals for the Ellipsoidal height difference and Orthometric height difference in Spreaded network

5.2. SKEWED DISTRIBUTION NETWORK

According to the skewed distribution network, η and ξ components of the deflection of vertical at the solution station A was found to be - 0° 0' 10.57460420" ± 0° 0' 6.16231994" and - 0° 0' 23.58654648 " ± 0° 0' 1.28286691 " respectively. And magnitude of the deflection of vertical was found to be 0° 0' 25.84854791" ± 0° 0' 3.00577546". Also the solution has four degree of freedom due to the six conditioned equations and two unknown parameters. The a priori variance is 1. The a posteriori variance is 0.13779109 and that indicates priori standard deviations are too small and observations have been adjusted more than predicted. Solution passed the two-tailed Chi-Squared Test at the 95% confidence level. In this network also residual and standard deviation for the orthometric height difference and ellipsoidal height difference were reached to mm level.

Standardized residuals can be used to identify and remove possible outlier observations. Standardized residual is the ratio between observation's actual fit in the adjustment and estimate of the strength. As well as the redundancy numbers for each adjusted orthometric height difference and ellipsoidal height difference observation were examined. In this case noise in the residual is small. The redundancy numbers of the adjusted observations are closer to 1 than 0.

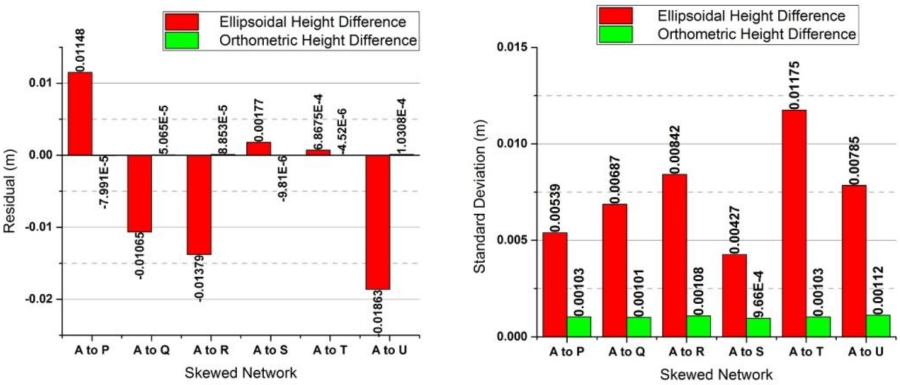

Figure 5: Residual and standard deviation for the ellipsoidal height difference and Orthometric height difference in skewed network

The following histogram of residuals suggests that the residuals (and hence the error terms) are normally distributed.

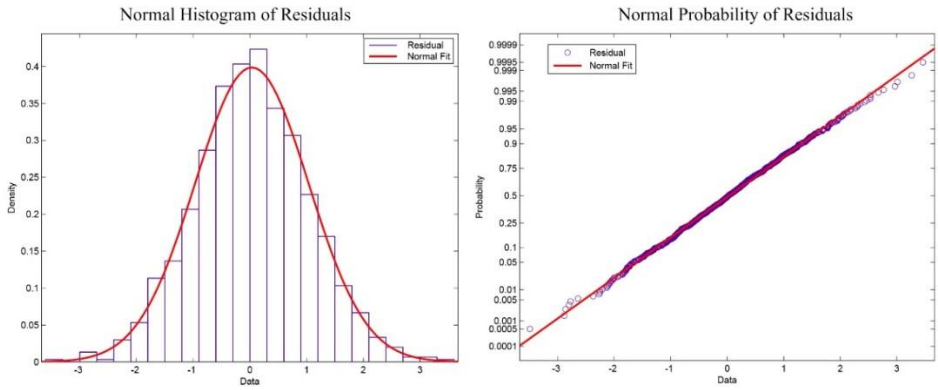

Figure 6: Normal histogram and normal probability of residuals for the Ellipsoidal height difference and orthometric height difference in skewed network

6. COMPARISON OF RESULTS FROM BOTH SPREADED AND SKEWED DISTRIBUTION NETWORK

Table 3: Comparison of the results

Method		The Deflection of vertical components		Magnitude of the deflection of vertical
		ξ (North-South)	η (East-West)	
GPS/Leveling	Spreaded Network	-25.0886" ± 3.7448"	-2.4909" ± 4.2971"	25.2120" ± 3.9050"
	Skewed Network	-23.5865" ± 1.2829"	-10.5746" ± 6.1623"	25.8485" ± 3.0058"
EGM 2008		-15.516"	-0.533"	-

According to the results of earlier studies by Tse and Bâki (2006), value of the east-west component was differed to some extent when compared with the values produced by other techniques. In the conclusion of their research, they explained the reason for that failure as the distribution geometry of the ancillary stations around the selected station. Therefore this study was tested the possibility of above statement by calculating the deflection of vertical at a same point by using two networks.

When comparing the deflection of vertical components obtained from spreaded distribution network with the values produced by the skewed distribution network, the north-south component was found to be approximately consistent with a small difference of 1.5021". But when considering the east-west component, it was differed to some extent with a considerable difference of 8.0837".

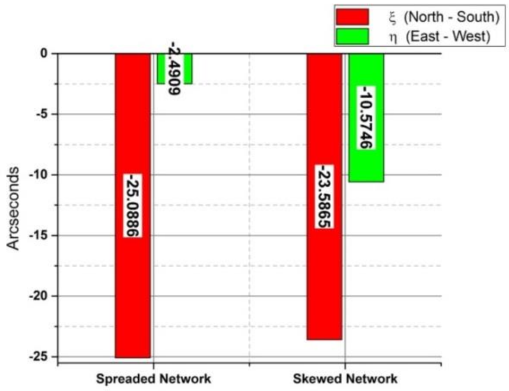

Figure 7: Graphical representation of the deflection of the vertical components results from two networks

When designing the two networks for field observations, ancillary stations in the skewed distribution network were designed as more biased towards the north-south direction. Therefore the effect of positions of the ancillary stations in the skewed distribution network is the most proximate reason for this value difference.

Nevertheless, magnitude of the deflection of vertical (ε) obtained from spreaded distribution network (ε = 25.21198004" ± 3.90504970") was not considerably differed from the value obtained from skewed distribution network (ε = 25.84854791" ± 3.00577546") as the difference was found to be 0.6365".

Obtained results were compared with the EGM2008 derived values for further confirmation. Although EGM2008 is the highest resolution global geopotential model available so far, it is not capable of representing the high-frequency components of Earth's gravity field. It represents gravity field quantities with wavelength approximately 10 arc minute, which equate to spatial resolution of 5 arc minutes. Therefore, it could be satisfied about the results which obtained from the GPS, leveling techniques. When comparing the obtained results of east-west component of the deflection of vertical in both networks with the EGM2008, EGM2008 values were much closer to the spreaded distribution network than the skewed distribution network. Here also east-west component of the deflection of vertical in skewed distribution network was more deviated with the other values. Therefore the effect of the geometry of the ancillary stations in the skewed distribution network might be one of the reasons for that deviation.

7. CONCLUSION AND RECOMMENDATIONS

GPS measurements and leveling networks were used to calculate the deflection of the vertical components of a fixed point in the area of Sabaragamuwa University. The study was carried out by using two distribution networks to check the effect of the geometry of the ancillary stations. According to the final results, the magnitude of the deflection of the vertical was nearly equal for both spreaded (ε = 25.21198004" ± 3.90504970") and skewed (ε = 25.84854791" ± 3.00577546") distribution networks. As stated by Bomford (1980), deflection of vertical can reach 20" in lowlands and up to 70" in rugged terrains. Therefore, obtained results seems reliable.

When comparing the results from the two networks with the EGM2008, there was a large deviation in the results of skewed distribution network. Therefore, distribution geometry of the ancillary stations around the selected station could be one of the factors affecting the final result.

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