Solution Analysis of Scale Factor in 3D Spatial Similarity Transformation

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ABSTRACT: Spatial similarity transformation can be mathematically performed through scaling, rotation and translation. From computational aspect, the scale factor of the similarity transformation can be solved either jointly with or separately from other two parameter sets. On some occasions, the true scale is seemed as the main focus while freeing the setting of the origins and poses of the coordinate systems. The quality of the scale factor estimation would depend on the types of measurement and the approaches used to reach the solution. The scale factor can be resolved under the concept of being the ratio of the corresponding distances between the two employed coordinate systems. As such, the related literatures under 3D similarity transformation theme have shown various ways of solving the scale factor based on the observed conditions and target error functions under the weighted least-squares adjustment technique. Thus, the aim of this study is to evaluate the quality of scale parameters solved through different methods. To this end, the theoretical precision and empirical accuracy realized in simulation data have been analyzed and the suggestions of attaining scale factor are made toward the practical applications.

1. INTRODUCTION

For 3D spatial similarity transformation, the scale factor is the ratio of the conjugate lengths (or distances). Although the scale factor is one of parameters in the transformation, it does not mean that one has to solve all the parameters to attain the scale factor. As proposed in many studies, the scale factor can be solved independently and linearly to fulfill the need of those tasks only involving the scale factor parameter or attempting to sequentially solve transformation parameters in a linear fashion. This study focuses on the direct solutions of scale factor on point basis. The comparisons of the equation forms and the solution quality are the main features concluded in this work.

2. LITERATURE REVIEW

The related scale factor solutions derived and reported from previous studies are reviewed as follows.

2.1 3D Spatial Similarity Transformation Flexible Formula with Scale Factor

Supposing that there are n points in two different coordinate systems, the 3D seven-parameter similarity transformation formula is given in a matrix form as Eq. (1):

$$s_i = t + \lambda R p_i , (i = 1, 2, 3, ..., n) \tag{1}$$

Where, $s_i = [X_i, Y_i, Z_i]^T$ (points in the target coordinate system), $t = [X_0, Y_0, Z_0]^T$ (the translation vector), λ (the scale factor), $R(\alpha,\beta,\gamma)$ (the rotation matrix), $p_i = [x_i, y_i, z_i]^T$ (points in the original coordinate system). It can be seen that the 3D seven-parameter transformation formula contains translation vector, rotation angles, and scale factor parameters. To eliminate the translation parameters, one can shift the origins of the coordinates to their corresponding centroids, thus with Eq. (2):

$$\Delta s_i = \lambda R \Delta p_i, (i = 1, 2, 3, ..., n) \tag{2}$$

with $\Delta s_i = s_i - \bar{s}$, $\Delta p_i = p_i - \bar{p}$, where $\bar{s}$ and $\bar{p}$ are the centroid coordinates of the two coordinate systems.

Through this process, the original equation (Eq. (1)) has been simplified to a formula without translation vector.

As the rotation matrix is an orthogonal one, Awange and Grafarend (2002) proposed the concept of anti-skew symmetric matrix C' as shown in Eq. (3):

$$R=(I-C')^{-1}(I+C'), \text{ where, } C' = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \tag{3}$$

Za'voti (2015) adopted the above approach and applied the centric coordinates to eliminate both the rotation matrix and the translation vector from the transformation and derived the quadratic form of scale factor formula as expressed in Eq. (4):

$$\lambda^2(\Delta p_i{}^T \Delta p_i) = (\Delta s_i{}^T \Delta s_i), (i = 1, 2, 3, ..., n) \tag{4}$$

It can be seen that there are indeed several forms of transformation, with complete set of parameters (Eq. (1)), partial parameters (Eq. (2)), or scale factor only (Eq. (3)) available. With focus on scale vector and its direct solution, Eqs. (2) and (3) are to be further studied for the associated solution forms.

2.2 Solutions of Scale Factor

The direct solutions of scale factor were derived by several authors and the details are given as follows:

Solution 1 of scale factor (Albertz –Kreiling, 1975)**:** The first step is to take the root of Eq. (4):

$$\lambda\sqrt{\Delta p_i{}^T \Delta p_i} = \sqrt{\Delta s_i{}^T \Delta s_i} \,, (i = 1, 2, 3, ..., n) \tag{5}$$

Adding all the observation using Eq. (4) to find the ratio of summed distances, thus λ_1 as shown in Eq. (6):

$$\lambda_1 = \frac{\sum_{i=1}^{n} \sqrt{\Delta s_i{}^T \Delta s_i}}{\sum_{i=1}^{n} \sqrt{\Delta p_i{}^T \Delta p_i}} \tag{6}$$

Eq. (6), the solution 1, has the mathematical meaning of being ~~is~~ the ratio of the sum of the conjugate distances between the two coordinate systems.

Solution 2_1 and solution 2_2 of scale factor (Horn, 1987)**:** Considering that there are always errors associated with observations, the residual vector for Eq. (2) is expressed as in Eq. (7):

$$\Delta v_i = \Delta s_i - \lambda R \Delta p_i, (i = 1, 2, 3, ..., n) \tag{7}$$

With the least-squares method by taking differentiation with respect to λ, one can obtain:

$$\lambda_{2\_1} = \frac{\sum_{i=1}^{n}(\Delta s_i{}^T R \Delta p_i)}{\sum_{i=1}^{n}(\Delta p_i{}^T \Delta p_i)} \tag{8}$$

Eq. (8), the solution 2_1, contains both rotation matrix and scale factor.

Horn (1987) proposed a symmetric expression of errors:

$$\Delta v_i = \frac{1}{\sqrt{\lambda}} \Delta s_i - \sqrt{\lambda} R \Delta p_i, (i = 1, 2, 3, ..., n) \tag{9}$$

When the errors expressed above, the least-squares solution can be reached as:

$$\lambda_{2\_2} = \sqrt{\frac{\sum_{i=1}^{n}(\Delta s_i{}^T \Delta s_i)}{\sum_{i=1}^{n}(\Delta p_i{}^T \Delta p_i)}} \tag{10}$$

Eq. (10), the solution 2_2, is estimated by taking the root of the ratio of summed square distances.

Solution 3 of scale factor (Za'voti and Kalmár, 2015)**:** Based on Eq. (5), the residual vector can be expressed as Eq. (11):

$$\Delta v_i = \sqrt{\Delta s_i{}^T \Delta s_i} - \lambda\sqrt{\Delta p_i{}^T \Delta p_i}, (i = 1, 2, 3, ..., n) \tag{11}$$

And the least-squares solution of scale factor can be obtained as:

$$\lambda_3 = \frac{\sum_{i=1}^{n} \sqrt{(\Delta s_i{}^T \Delta s_i)(\Delta p_i{}^T \Delta p_i)}}{\sum_{i=1}^{n}(\Delta p_i{}^T \Delta p_i)} \tag{12}$$

Eq. (12), the solution 3, has both target and original components in numerator, as that of solution 2_1, yet with scale factor only.

Solution 4_1 and solution 4_2 of scale factor: Since the scale factor is conceptually the ratio of the conjugate lengths (or distances). With such, the scale factor solution can also be found from the individual scale observations either with equal weight (Eq. (13), solution 4_1) or weighted (Eq. (14), solution 4_2) approaches.

$$\lambda_{4\_1} = \left(\sum_{i=1}^{n} \frac{\sqrt{\Delta s_i{}^T \Delta s_i}}{\sqrt{\Delta p_i{}^T \Delta p_i}}\right)/n \tag{13}$$

$$\lambda_{4\_2} = \left(\sum_{i=1}^{n} P_i \frac{\sqrt{\Delta s_i{}^T \Delta s_i}}{\sqrt{\Delta p_i{}^T \Delta p_i}}\right)/\sum_{i=1}^{n} P_i, \text{ with } P_i \text{ the weight of } \frac{\sqrt{\Delta s_i{}^T \Delta s_i}}{\sqrt{\Delta p_i{}^T \Delta p_i}} \tag{14}$$

3. STUDY OF METHOD

This study discusses and compares the similarities and differences of each solution of the scale factor based on the point observations with centric coordinates. And then the evaluation of solutions is followed in the experiments. The overall comparisons of scale factor solutions are given in section 3.1, while section 3.2 analyzes whether the rotation matrix affects the solution, and section 3.3 proves the equivalence of some solutions.

3.1 Comparisons of Scale Factor Solutions

Based on Eqs. (1) ~ (3), it can be stated that the root of each solution is either derived from coordinate transformation, the conjugate distances, or the scale solution itself. Figure 1 provides the scale factor solution scheme to highlight how they were formulated.
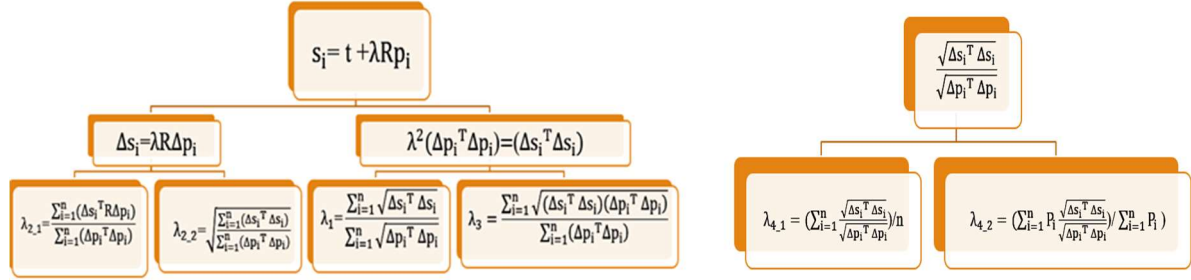

Figure 1. Flow chart of scale factor solution

Observing each solution, only solution_2 contains a rotation matrix. Solution 2_1, solution 2_2, and solution_3 were all derived from the least-squares method, so they should have the best precision theoretically.

3.2 Effect of Rotation Matrix

The rotation matrix is an orthogonal matrix, which does not affect the length. As a linear mapping (transformation matrix), the orthogonal matrix can keep the distance constant, so it is also called equidistant structure. Rotation in geometry and linear algebra is a motion describing a rigid body around a fixed point (Figure 2), a transformation in a plane or space, the rotation and the transformation mentioned above are equidistant, the distance does not change after the transformation. The proof of equidistant structure is given as follows.

The following 3D coordinate system shows that the original coordinate system is rotated to another coordinate system, expressed in matrix form as:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{15}$$

When calculating the distance, the matrix must be multiplied by its own transposed matrix. It is known that the rotation matrix is an orthogonal matrix, so the transposed matrix is equivalent to the inverse matrix. The transposed rotation matrix is multiplied by the original one to attain the identity matrix, as shown in Eq. (16), so the rotation matrix is proved to not to affect the distance through the transformation.

$$[x'\ y'\ z'] \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [x\ y\ z] R^T R \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [x\ y\ z] I \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{16}$$
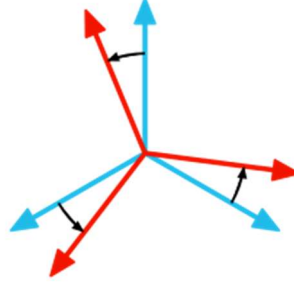

Figure 2. 3D space rotation does not affect the distance

3.3 The Equivalence of The Solutions Derived from The Least-Squares Method

It is known that λ_{2_1}, λ_{2_2}, and λ_3 are all obtained by the least-squares method, except that the target functions are different, with λ_{2_1} derived from $\Delta v_i = \Delta s_i - \lambda R \Delta p_i$, λ_3 from $\Delta v_i = \sqrt{\Delta s_i^T \Delta s_i} - \lambda \sqrt{\Delta p_i^T \Delta p_i}$, and λ_{2_2} from $\Delta v_i = \frac{1}{\sqrt{\lambda}} \Delta s_i - \sqrt{\lambda} R \Delta p_i$, to find the best solution of scale factor. As $\sqrt{(\Delta s_i^T \Delta s_i)(\Delta p_i^T \Delta p_i)}$ is equal to $\Delta s_i^T \Delta p_i$, so $\sum_{i=1}^{n}(\Delta s_i^T R \Delta p_i)$ and $\sum_{i=1}^{n} \sqrt{(\Delta s_i^T \Delta s_i)(\Delta p_i^T \Delta p_i)}$ are actually identical.

Solution 2_1, solution 2_2, which were both derived from $\Delta v_i = \Delta s_i - \lambda R \Delta p_i$. Therefore, they should be the same, and the optimal property of the estimator described in the mathematical statistics is mainly that the estimator should have unbiasedness, consistency, validity, and processing with random error according to the least-squares method.

The above analysis concludes that the solution 2_1, solution 2_2, and solution 3 are equivalent and all result from least-squares method.

3.4 Precision Evaluation Method

As for the precision evaluation, the theoretical precision is evaluated by the error propagation. The variance propagation method is used to find the variance of the six kinds of scale factor solutions listed above, it is the most direct and effective way to offer the solution quality since the partial derivative components of the point coordinates before and after the transformation can be taken through Eq. (17), taking the λ_1 as the example:

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$$\sigma_{\lambda_1}{}^2 = \Sigma_{\lambda_1\lambda_1} = \begin{bmatrix} \partial\lambda_1/\partial x_1 \\ \partial\lambda_1/\partial y_1 \\ \partial\lambda_1/\partial z_1 \\ \cdots \\ \partial\lambda_1/\partial x_n \\ \partial\lambda_1/\partial y_n \\ \partial\lambda_1/\partial z_n \\ \cdots \\ \partial\lambda_1/\partial x'_1 \\ \partial\lambda_1/\partial y'_1 \\ \partial\lambda_1/\partial z'_1 \\ \cdots \\ \partial\lambda_1/\partial x'_n \\ \partial\lambda_1/\partial y'_n \\ \partial\lambda_1/\partial z'_n \end{bmatrix}^T \begin{bmatrix} \sigma x_1{}^2 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \sigma z'_n{}^2 \end{bmatrix} \begin{bmatrix} \partial\lambda_1/\partial x_1 \\ \partial\lambda_1/\partial y_1 \\ \partial\lambda_1/\partial z_1 \\ \cdots \\ \partial\lambda_1/\partial x_n \\ \partial\lambda_1/\partial y_n \\ \partial\lambda_1/\partial z_n \\ \cdots \\ \partial\lambda_1/\partial x'_1 \\ \partial\lambda_1/\partial y'_1 \\ \partial\lambda_1/\partial z'_1 \\ \cdots \\ \partial\lambda_1/\partial x'_n \\ \partial\lambda_1/\partial y'_n \\ \partial\lambda_1/\partial z'_n \end{bmatrix} \tag{17}$$

4. EXPERIMENT AND ANALYSIS

The error propagation mentioned above can directly consider the error on the points before and after the conversion. Usually, the discrepancy (Figure 3) of transformation needs to be absorbed by the residual vector together with the estimated parameter.
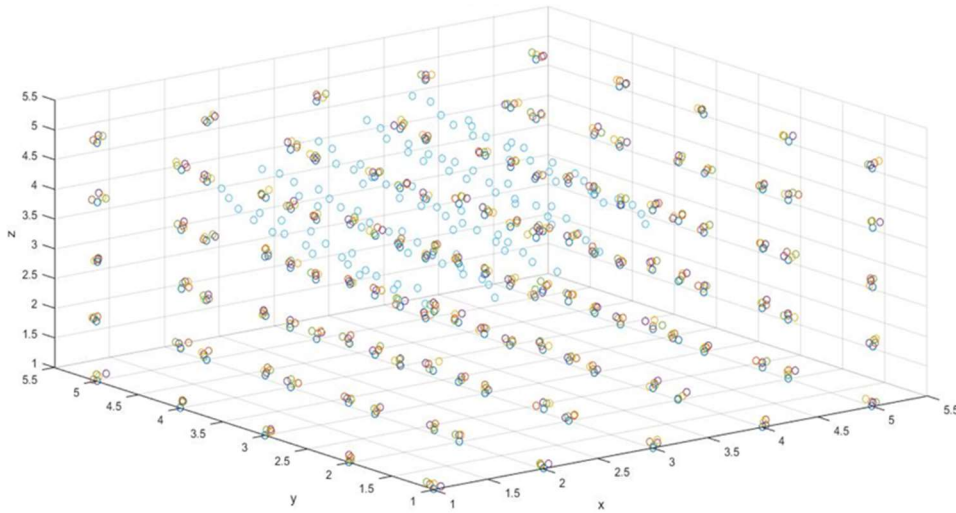

Figure 3. 3D space conversion affected by errors

The experiments are mainly based on simulated data sets as stated below. Starting with seven-parameter transformation, the translation, rotation matrix, scale factor, and a set of points are hypothesized to serve as the true values. The purpose of simulating three experiments, shown as below, is to observe the influences of modulating number of points, level of accuracy, and magnitude of scale factor.

Test1: There are three points (1,1,1), (2,2,2), (3,3,3) before the conversion. These three points are rotated and scaled (magnification is 0.5), the performances of six solutions are analyzed.

Case1: Given a random error with an overall standard deviation of 0.0005 for points in both coordinate systems, the quality of six solutions estimated is shown in Table 1.

Table 1. Values and precision of each scale factor solution

	Solution 1	Solution 2_1	Solution 2_2	Solution 3	Solution 4_1	Solution 4_2
λ	0.5014	0.5009	0.5009	0.5009	1.4020	0.5335
σ_λ	2.6364e-04	2.2828e-04	2.2828e-04	2.2828e-04	0.5896	0.0167

Case2: Given random errors with overall standard deviation of 0.0005 and 0.0001 to the points in original and target coordinate systems, respectively, the quality of six solutions estimated is shown in Table 2.

Table 2. Values and precision of each scale factor solution

	Solution 1	Solution 2_1	Solution 2_2	Solution 3	Solution 4_1	Solution 4_2
λ	0.5015	0.5010	0.5010	0.5010	1.2830	0.5368
σ_λ	1.2721e-04	1.1009e-04	1.1009e-04	1.1009e-04	0.5299	0.0151

Case3: Given a random error with an overall standard deviation of 0.0001 and 0.0005 to the points in original and target coordinate systems, respectively, the quality of six solutions estimated is shown in Table 3.

Table 3. Values and precision of each scale factor solution

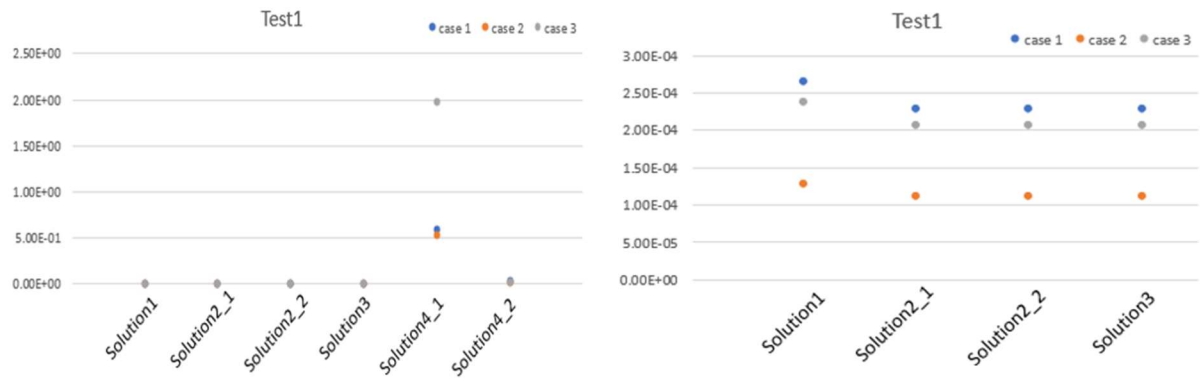|	Solution 1	Solution 2_1	Solution 2_2	Solution 3	Solution 4_1	Solution 4_2
λ	0.5016	0.5011	0.5011	0.5011	3.8194	0.5125
σ_λ	2.3690e-04	2.0515e-04	2.0515e-04	2.0515e-04	1.9688	0.0061

Figure 4. Precision of each solution in test1

Analysis in test1: The best solutions, based on the standard deviation of the scale parameter, are found in solution 2_1, solution 2_2, and solution 3, derived based on least-squares principle. The secondary best solution is solution 1 and followed by solution 4_2. The solution 4_1 behaves the worst. The empirical accuracy given by the error the estimated parameter also suggests the very same trend. Regardless of solution types, the larger the coordinate uncertainty, the less precise the scale factor obtained.

Test2: Compared with test1, more points have been deployed in test2. Gridded points (1,1,1) to (3,3,3), 27 points in total, are rotated and scaled (magnification is 0.5), the performances of six solutions are analyzed.

Case1: Given a random error with an overall standard deviation of 0.0005 for points in both coordinate systems, the quality of six solutions estimated is shown in Table 4.

Table 4. Values and precision of each scale factor solution

	Solution 1	Solution 2_1	Solution 2_2	Solution 3	Solution 4_1	Solution 4_2
λ	0.5000	0.4999	0.4999	0.4999	0.5484	0.5038
σ_λ	7.8818e-05	7.6069e-05	7.6069e-05	7.6069e-05	0.0384	0.0013

Case2: Given random errors with overall standard deviation of 0.0005 and 0.0001 to the points in original and target coordinate systems, respectively, the quality of six solutions estimated is shown in Table 5.

Table 5. Values and precision of each scale factor solution

	Solution 1	Solution 2_1	Solution 2_2	Solution 3	Solution 4_1	Solution 4_2
λ	0.5000	0.5000	0.5000	0.5000	0.4949	0.4907
σ_λ	3.7966e-05	3.6643e-05	3.6643e-05	3.6643e-05	0.0161	0.0501

Case3: Given a random error with an overall standard deviation of 0.0001 and 0.0005 to the points in original and target coordinate systems, respectively, the quality of six solutions estimated is shown in Table 6.

Table 6. Values and precision of each scale factor solution

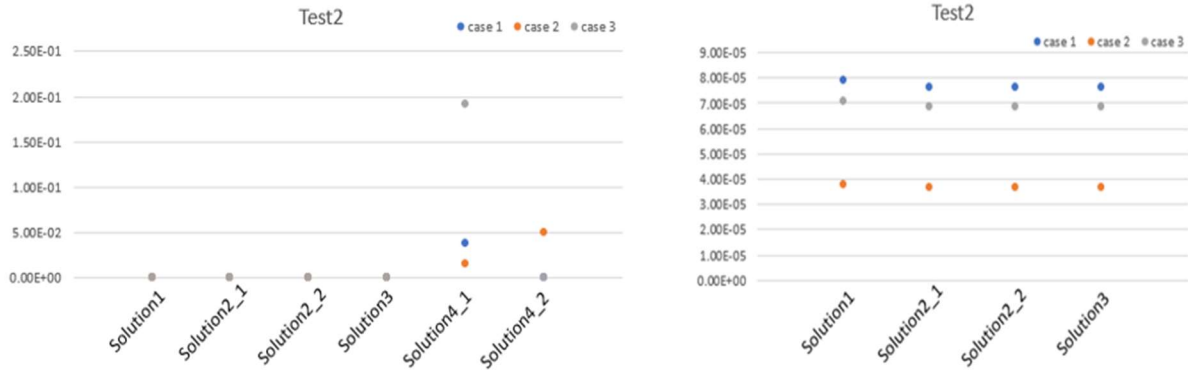|	Solution 1	Solution 2_1	Solution 2_2	Solution 3	Solution 4_1	Solution 4_2
λ	0.5000	0.5000	0.5000	0.5000	0.8165	0.5009
σ_λ	7.0851e-05	6.8380e-05	6.8380e-05	6.8380e-05	0.1922	5.4710e-04

Figure 5. Precision of each solution in test2

Analysis in test2: All the results nearly resemble the test1. The best solutions come from least-squares derived estimators. The only difference is that the parameter estimation benefits from more participating points, strengthening the solutions. Comparing the tables in test1 and test2, the gain of precision is three by virtue of 9 times observations.

Test3: To illustrate the scaling effect, three points (1,1,1), (2,2,2), (3,3,3) are rotated and scaled (magnification is 2), the performances of six solutions are analyzed.

Case1: Given a random error with an overall standard deviation of 0.0005 for points in both coordinate systems, the quality of six solutions estimated is shown in Table 7.

Table 7. Values and precision of each scale factor solution

	Solution 1	Solution 2_1	Solution 2_2	Solution 3	Solution 4_1	Solution 4_2
λ	1.9994	1.9995	1.9995	1.9995	1.6781	1.8923
σ_λ	5.2673e-04	4.5621e-04	4.5621e-04	4.5621e-04	0.6250	0.2083

Case2: Given random errors with overall standard deviation of 0.0005 and 0.0001 to the points in original and target coordinate systems, respectively, the quality of six solutions estimated is shown in Table 8.

Table 8. Values and precision of each scale factor solution

	Solution 1	Solution 2_1	Solution 2_2	Solution 3	Solution 4_1	Solution 4_2
λ	2.0002	2.0004	2.0004	2.0004	1.4380	1.8130
σ_λ	4.7383e-04	4.1044e-04	4.1044e-04	4.1044e-04	0.1095	0.0365

Case3: Given a random error with an overall standard deviation of 0.0001 and 0.0005 to the points in original and target coordinate systems, respectively, the quality of six solutions estimated is shown in Table 9.

Table 9. Values and precision of each scale factor solution

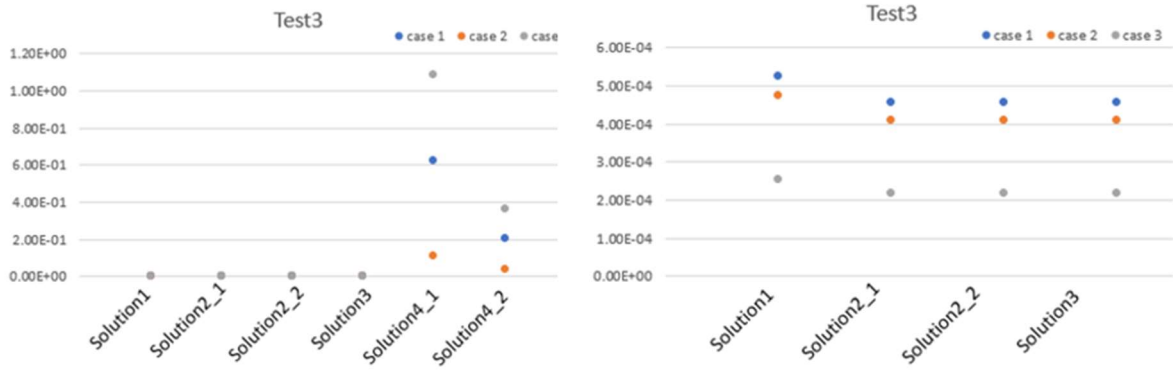|	Solution 1	Solution 2_1	Solution 2_2	Solution 3	Solution 4_1	Solution 4_2
λ	1.9996	1.9996	1.9996	1.9996	1.9851	1.9948
σ_λ	2.5382e-04	2.1983e-04	2.1983e-04	2.1983e-04	1.0886	0.3628

Figure 6. Precision of each solution in test3

Analysis in test3: Scaling up or down would affect the estimate with the errors varies between two coordinate systems. It shows that in scaling up case (Test 3: the magnitude of scale factor is 2), if the random errors in the original coordinate system are larger than that in target one, the scale factor solution is worsened due to the addition of enlarged point errors in the original coordinate system to the overall estimation, as shown in case2 vs. case3.
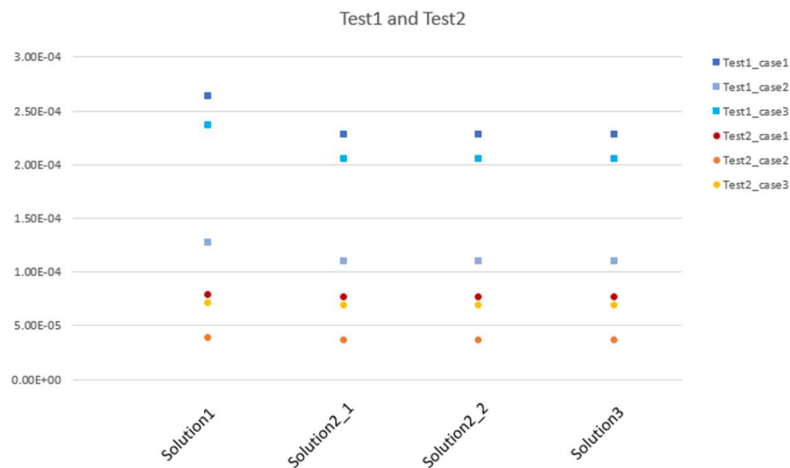

Figure 7. Precision of each solution in test1 and test2

Analysis in terms of precision: λ_{2_1}, λ_{2_2} and λ_3 are derived by the least-squares method, so it can be understood that they have the best precision. On the other hand, the precision of λ_{4_2} with weighting average, can be expected to be higher than that of λ_{4_1}, unweighted estimate. λ_1 is the ratio of the sum of the distances between the two coordinate systems, it is still more effective and closer to the definition of the scale than λ_{4_1} and λ_{4_2}. Yet, it remains interesting to quantify how much difference between the solution 4_2 and other least-squares derived solutions, since the former one has been so often considered in routine tasks.

5. CONCLUSIONS AND FUTURE WORK

In the direct solution model of the scale factor, this study explores and analyzes six different solutions. Although the most basic definition of the scale factor is the ratio of conjugate lengths, there are still different scale factor solutions under different preconditions. Through the mathematical analysis and numerical simulations, the three scale factors obtained by least-squares method have the same value and with the best precision. The number of points and the level of error all contribute to the scale factor solution as expected. Yet, the magnitude of scale factor, scaled up or scaled down, combined with the varied precision of points in two coordinates complicates the quality evaluation and makes itself appealing for further study. Last but not least, although solution 2_1 is among the best ones, the required rotation matrix makes the application of it in a more restricted way, unlike solution 2_2 and solution 3 purely basing on coordinates.

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